where $V^{(e)}$ is the volume of the tetrahedron element (e) given by

$$V^{(e)} = \frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix}$$

and

$$a_1 = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}, \quad b_1 = -\begin{vmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}$$

$$c_1 = -\begin{vmatrix} x_2 & 1 & z_2 \\ x_3 & 1 & z_3 \\ x_4 & 1 & z_4 \end{vmatrix}, \quad d_1 = -\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$$

The other constants are obtained by the interchange of the subscripts in the order 1234. Substituting (8.102) into (8.100) we get

$$\begin{aligned} u^{(e)}(x, y, z) &= N_1(x, y, z)\, u_1 + N_2(x, y, z)\, u_2 + N_3(x, y, z) u_3 \\ &\quad + N_4(x, y, z) u_4 \\ &= \mathbf{N}^{(e)} \boldsymbol{\phi}^{(e)} \end{aligned} \tag{8.103}$$

where

$$\mathbf{N}^{(e)} = [N_1\ N_2\ N_3\ N_4]$$

$$\boldsymbol{\phi}^{(e)} = [u_1\ u_2\ u_3\ u_4]^T$$

$$N_i = \frac{1}{6V^{(e)}}(a_i + b_i x + c_i y + d_i z), \qquad i = 1(1)4 \tag{8.104}$$

We find that the shape functions satisfy

$$N_i(x_j, y_j\, z_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

The location of any point $P(x, y, z) \in (e)$ can be defined in terms of the volume coordinates (L_1, L_2, L_3, L_4) given by

$$L_1 = \frac{\text{vol P234}}{V^{(e)}}, \qquad L_2 = \frac{\text{vol P341}}{V^{(e)}}$$

$$L_3 = \frac{\text{vol P412}}{V^{(e)}}, \qquad L_4 = \frac{\text{vol P123}}{V^{(e)}} \tag{8.105}$$

The *volume coordinate* system is also known as local or *natural* coordinate system. Since (see Figure 8.6(b))

$$\text{vol P234} + \text{vol P341} + \text{vol P412} + \text{vol P123} = V^{(e)}$$

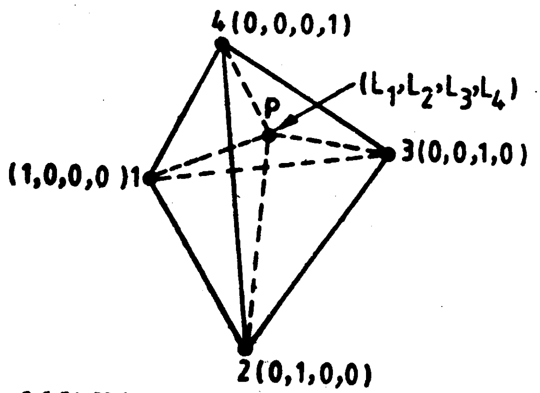

Fig. 8.6 (b) Volume coordinate for a tetrahedron element

we can write

$$L_1+L_2+L_3+L_4=1 \tag{8.106}$$

Further we have

$$L_i=N_i \qquad i=1(1)4 \tag{8.107}$$

and the coordinates of the vertices 1, 2, 3 and 4 in terms of the local coordinates become (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), respectively. The cartesian and local coordinates are related by

$$\begin{aligned} x&=x_1N_1+x_2N_2+x_3N_3+x_4N_4\\ y&=y_1N_1+y_2N_2+y_3N_3+y_4N_4\\ z&=z_1N_1+z_2N_2+z_3N_3+z_4N_4 \end{aligned} \tag{8.108}$$

Using (8.106) and (8.105), we may express the local coordinates in terms of of the cartesian coordinates as

$$\begin{bmatrix} L_1\\ L_2\\ L_3\\ L_4 \end{bmatrix} \quad \begin{bmatrix} a_1 & b_1 & c_1 & d_1\\ a_2 & b_2 & c_2 & d_2\\ a_3 & b_3 & c_3 & d_3\\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \begin{bmatrix} 1\\ x\\ y\\ z \end{bmatrix} \tag{8.109}$$

The differentiation and integration formulas corresponding to (8.70) and (8.72) are given by

$$\begin{aligned} \frac{\partial N_i}{\partial x}&=\sum_{j=1}^{4}\frac{\partial N_i}{\partial L_j}\frac{\partial L_j}{\partial x}\\ \frac{\partial N_i}{\partial y}&=\sum_{j=1}^{4}\frac{\partial N_i}{\partial L_j}\frac{\partial L_j}{\partial y}\\ \frac{\partial N_i}{\partial z}&=\sum_{j=1}^{4}\frac{\partial N_i}{\partial L_j}\frac{\partial L_j}{\partial z} \end{aligned} \tag{8.110}$$

$$\int_{V^{(e)}} L_1^r L_2^s L_3^t L_4^q \, dV = \frac{r!\, s!\, t!\, q!\, 6V^{(e)}}{(r+s+t+q+3)!} \tag{8.111}$$

where

$$\frac{\partial L_j}{\partial x} = \frac{b_j}{6V^{(e)}}, \quad \frac{\partial L_j}{\partial y} = \frac{c_j}{6V^{(e)}}, \quad \frac{\partial L_j}{\partial z} = \frac{d_j}{6V^{(e)}}$$

and r, s, t, q are positive integers.

Higher degree piecewise polynomial

The pth degree piecewise polynomial in three space variables x, y and z may be written as

$$u^{(e)}(x, y, z) \sum_{r+s+t=0}^{p} a_{rst} x^r y^s z^t \tag{8.112}$$

where the parameters a_{rst} are determined by using the interpolation conditions. We choose $\frac{1}{6}(p+1)(p+2)(p+3)$ nodes on the tetrahedron element (e). The equation (8.112) in terms of the nodal values u_i is of the form

$$u^{(e)} = \sum_{i=1}^{N} N_i(x, y, z) u_i \tag{8.113}$$

where N is the number of nodes in the element. The shape functions N_i of (8.113) can be generated using the natural coordinates L_1, L_2, L_3, and L_4. The distribution of nodes for the piecewise quadratic ($p=2$) and cubic ($p=3$) polynomials are shown in Figures 8.6(c) and 8.6(d).

Quadratic Lagrange polynomial

The piecewise quadratic polynomial becomes

$$u^{(e)}(x, y, z) = \sum_{i=1}^{10} N_i u_i \tag{8.114}$$

where (see Fig. 8.6(c))

$$N_i = L_i(2L_i - 1), \quad i = 1, 2, 3, 4$$

$$N_5 = 4L_1L_2, \quad N_6 = 4L_2L_4, \quad N_7 = 4L_1L_3$$

$$N_8 = 4L_1L_4, \quad N_9 = 4L_2L_4, \quad N_{10} = 4L_3L_4$$

Cubic Lagrange polynomial

The piecewise cubic polynomial becomes

$$u^{(e)}(x, y, z) = \sum_{i=1}^{20} N_i u_i \tag{8.115}$$

where (see Fig. 8.6(d)) for corner nodes

$$N_i = \tfrac{1}{2}L_i(3L_i - 1)(3L_i - 2), \quad i = 1, 2, 3, 4$$

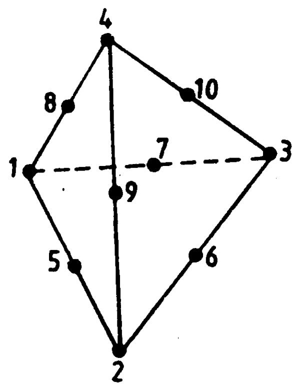

Fig. 8.6 (c) Quadratic element with ten nodes

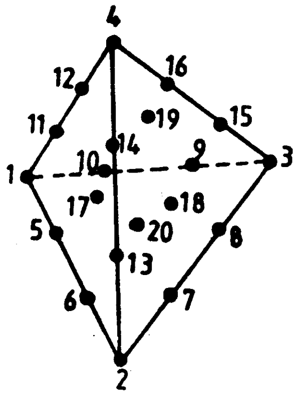

Fig. 8.6 (d) Cubic element with twenty nodes

for one-third nodes of edges

$$N_5 = \tfrac{9}{2} L_1L_2(3L_1 - 1), \qquad N_6 = \tfrac{9}{2} L_1L_2(3L_2 - 1)$$

$$N_7 = \tfrac{9}{2} L_2L_3(3L_2 - 1), \qquad N_8 = \tfrac{9}{2} L_2L_3(3L_3 - 1), \text{ etc}$$

and for midface nodes

$$N_{17} = 27L_1L_2L_4, \qquad N_{18} = 27L_2L_3L_4$$

$$N_{19} = 27L_1L_3L_4, \qquad N_{20} = 27L_1L_2L_3 \tag{8.116}$$

8.4.6 Hexahedron element

The three dimensional domain $\mathcal{R}$ can also be discretized using the hexahedron elements with four quadrilateral faces. We choose an arbitrary hexahedron element (e) with eight nodal points (x_i, y_i, z_i), $i = 1(1)8$ at the corners. The function value at the node i is represented by u_i. We take the origin of the local coordinates (ξ, η, ζ) at the point of intersection of the lines joining the mid-points of the opposite faces of the hexahedron and define the sides by $\xi = \pm 1$, $\eta = \pm 1$ and $\zeta = \pm 1$ as shown in Figure 8.7(a). The transformation

$$x = \sum_{i=1}^{8} N_i x_i, \qquad y = \sum_{i=1}^{8} N_i y_i,$$

$$z = \sum_{i=1}^{8} N_i z_i \tag{8.117}$$

where

$$N_i = \tfrac{1}{8}(1 + \xi_i\xi)(1 + \eta_i\eta)(1 + \zeta_i\zeta) \tag{8.118}$$

$$[\xi_1\ \xi_2\ \xi_3\ \xi_4\ \xi_5\ \xi_6\ \xi_7\ \xi_8]^T = [-1\ \ 1\ \ 1\ \ -1\ \ -1\ \ 1\ \ 1\ \ -1]^T$$

$$[\eta_1\ \eta_2\ \eta_3\ \eta_4\ \eta_5\ \eta_6\ \eta_7\ \eta_8]^T = [-1\ \ -1\ \ 1\ \ 1\ \ -1\ \ -1\ \ 1\ \ 1]^T$$

$$[\zeta_1\ \zeta_2\ \zeta_3\ \zeta_4\ \zeta_5\ \zeta_6\ \zeta_7\ \zeta_8]^T = [-1\ \ -1\ \ -1\ \ -1\ \ 1\ \ 1\ \ 1\ \ 1]^T$$

transforms the hexahedron element (e) into a cube $|\xi| = 1$, $|\eta| = 1$, $|\zeta| = 1$ as shown in Figure 8.7(b).